Turbo Joint Source-Channel Coding of Non-Uniform Memoryless Sources in the Bandwidth-Limited Regime

Idoia Ochoa, *Student Member, IEEE*, Pedro Crespo, *Senior Member, IEEE*,
Javier Del Ser, *Member, IEEE*, and Mikel Hernaez, *Student Member, IEEE*

***Abstract*—This letter proposes a novel one-layer coding/shaping scheme with single-level codes and sigma-mapping for the bandwidth-limited regime. Specifically, we consider non-uniform memoryless sources sent over AWGN channels. At the transmitter, binary data are encoded by a Turbo code composed of two identical RSC (*Recursive Systematic Convolutional*) encoders. The encoded bits are randomly interleaved and modulated before entering the sigma-mapper. The modulation employed in this system follows the unequal energy allocation scheme first introduced in [1]. The receiver consists of an iterative demapping/decoding algorithm, which incorporates the *a priori* probabilities of the source symbols. To the authors' knowledge, work in this area has only been done for the power-limited regime. In particular, the authors in [2] proposed a scheme based on a Turbo code with RSC encoders and unequal energy allocation. Therefore, it is reasonable to compare the performance – with respect to the Shannon limit – of our proposed bandwidth-limited regime scheme with this former power-limited regime scheme. Simulation results show that our performance is as good or slightly better than that of the system in [2].**

***Index Terms*—Sigma-mapping, non-uniform memoryless sources, turbo codes, bandwidth-limited regime.**

I. INTRODUCTION

We consider the transmission of the information generated by a non-uniform memoryless source with probability distribution $(p_0, p_1 = 1 - p_0)$ over the AWGN channel. Shannon's Separation Theorem states that source coding and channel coding can be carried out in isolation. Thus, the standard approach has been to separate the encoding process in two parts: first, a source encoder capable of compressing the source up to its theoretical limit (given by its entropy $H(p_0)$), and second, a capacity-achieving channel code. Consequently, the E_{so} (average energy per source symbol) lower limit is given by

$$\frac{E_{so}}{N_0} > \frac{2^\rho - 1}{R}, \tag{1}$$

where N_0 is the noise variance per two dimension (2D), ρ is the spectral efficiency in bits per 2D, and $R = \rho/H(p_0)$ is the transmission rate in source symbols per complex channel symbol (or per two real channel symbols). We will refer to joint source-channel coding schemes working at $R \geq 2$ as coding schemes operating in the bandwidth-limited regime; otherwise it is said that they are operating in the power-limited regime.

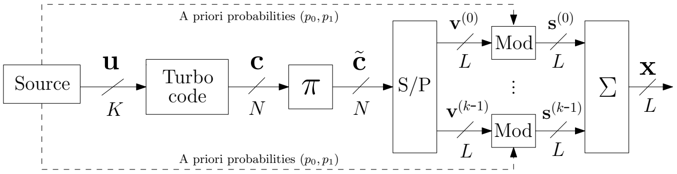

Fig. 1. Proposed Turbo encoding, modulation and sigma-mapping scheme.

However, when complexity is an issue, the overall performance can be improved if the tasks of source and channel coding are blended together by means of a joint source-channel encoder. In this way, the joint decoder can use some of the inherent redundancy of the source to alleviate the requirements of the channel encoder. For this reason, in the scheme proposed in this letter source and channel coding are jointly performed by a Turbo encoder operating at the bandwidth-limited regime in conjunction with a power-allocation strategy to achieve a shaping gain. To the best of our knowledge, previous work on joint source-channel coding for non-uniform memoryless sources has only been done for the power-limited regime. As to mention, in [3] they use LDPC codes, while [4], [5] employ non-systematic Turbo codes.

The remainder of the paper is organized as follows. In Section II we describe the proposed system, whereas Section III details the decoding process. Simulation results are presented in Section IV, and finally Section V concludes the paper.

II. PROPOSED SYSTEM

The proposed transmission scheme is shown in Fig. 1. The sequence $\mathbf{u}$ of length K, which is generated by a non-uniform memoryless binary source with *a priori* probabilities p_0 and p_1, is encoded through a Turbo code of rate[1] $R_c = 1/n$, producing the encoded sequence $\mathbf{c}$ of length $N = nK$ which is next processed through an interleaver π. The interleaved version $\tilde{\mathbf{c}}$ of $\mathbf{c}$ is converted by a serial-to-parallel converter into k sequences $\mathbf{v}^{(i)}$ of length L, where $0 \leq i \leq k$-1. Then, the modulator assigns different amplitudes (and consequently, different energies) to the encoded symbols, yielding k non-binary sequences $\mathbf{s}^{(i)}$. Following the scheme in [2], the symbols associated to systematic bits 1 and 0 are, respectively, $+\sqrt{E_1^s} = +\sqrt{p_0/p_1}$ and $-\sqrt{E_0^s} = -\sqrt{p_1/p_0}$, where $p_1 E_1^s + p_0 E_0^s = 1$. For the parity bits, the associated symbols depend on the value of both the parity and the input bit of the underlying RSC code of the Turbo encoder. Notice

Manuscript received December 22, 2009. The associate editor coordinating the review of this letter and approving it for publication was Z. Yan.

I. Ochoa, P. M. Crespo, and M. Hernaez are with the Centro de Estudios e Investigaciones Técnicas de Gipuzkoa (CEIT) and TECNUN (University of Navarra), 20009 San Sebastian, Spain (e-mail: {iochoa, pcrespo, mhernaez}@ceit.es).

J. Del Ser is with TECNALIA-TELECOM, 48170 Zamudio, Spain (e-mail: jdelser@robotiker.es).

Digital Object Identifier 10.1109/LCOMM.2010.04.092462

---

[1] Notice that R_c (*code rate*) is not necessary equal to R (*transmission rate*) defined in expression (1).

that the input sequence of one RSC code corresponds to the systematic bits, while the input of the other one corresponds to the interleaved version of the source sequence. The amplitudes of parity bits 1 and 0 are given by $+\sqrt{E_i^p}$ and $-\sqrt{E_i^p}$, respectively, where the subindex i will take the value 0 or 1 depending on the value of the associated input bit. E_1^p is set to $(1-\theta)/p_1$, and E_0^p to θ/p_0, where $p_1 E_1^p + p_0 E_0^p = 1$. The value of θ $(0 \leq \theta \leq 1)$ is chosen by simulation so as to minimize the probability of error at the receiver, which is usually achieved when $\theta = 0.5$ [2]. Thus, systematic bits can be mapped to 2 different values, while parity bits to 4 different values.

These modulated sequences enter the sigma-mapper Σ [6], which generates a single signal sequence $\mathbf{x}$ of length L. It should be mentioned that the reason to have used a sigma-mapper instead of any other alternative mapping, is that it blends together perfectly with the unequal power allocation technique. At time t, the output of the sigma-mapper is denoted as $x_t = \phi_\Sigma(\mathbf{s}_t)$, where $\mathbf{s}_t = (s_t^{(0)}, ..., s_t^{(k-1)})$ and $\phi_\Sigma(\mathbf{s}_t) \triangleq \sum_{i=0}^{k-1} \alpha_i s_t^{(i)}$. The sigma-mapper utilized in our scheme is known as *Type-I* [2], since $\alpha_i = \alpha \; \forall i \in \{0, \ldots, k\text{-}1\}$. The value of α is chosen to satisfy $E_{x_t} = E_c$, and it can be shown that α should be set to $\sqrt{E_c/k}$. The received sequence $\mathbf{y}$ at destination is a version of $\mathbf{x}$ corrupted by Additive White Gaussian Noise (AWGN).

III. DECODING PROCESS

The receiver shown in Fig. 2 iterates between the sigma-demapper (labeled as Σ^{-1}) and the Turbo decoder. The sigma-demapper is based on a SISO (*Soft-Input Soft-Output*) demapping algorithm, and the Turbo decoder is implemented by the Sum-Product Algorithm (SPA) applied to the Factor Graph (FG) that describes the Turbo code. The decoding procedure allows for the successive exchange of *extrinsic* probabilities between the sigma-demapper and the Turbo decoder, which iteratively refines the *a posteriori* probabilities of the original source symbols. The sigma-demapper processes the sequence $\mathbf{y}$ and estimates the probabilities of the k superimposed symbols contained in each channel symbol y_t $(1 \leq t \leq L)$. If the symbol corresponds to a systematic bit, the sigma-demapper generates the *extrinsic* probability $P_v^{(e)}(m)$, where $m \in \{0, 1\}$. Accordingly, if the symbol represents a parity bit, the demapper calculates the probabilities $P_v^{(e)}(m|\textit{input bit})$, where the input bit can take values 0 and 1.

Hereafter, the superscript $(initial)$ represents the non-uniformity of the source. Thus, for the systematic bits, the probabilities $P_v^{(initial)}(m)$ will be p_0 for $m = 0$ and p_1 for $m = 1$, while for the parity bits they are always set to 0.5. The superscript (a) refers to the *a priori* probabilities coming from the Turbo decoder. In the first iteration they are equal to 0.5. Although the initial probabilities of the source can be estimated by both the encoder and the decoder [2], we will assume they are known in order to achieve optimum performance.

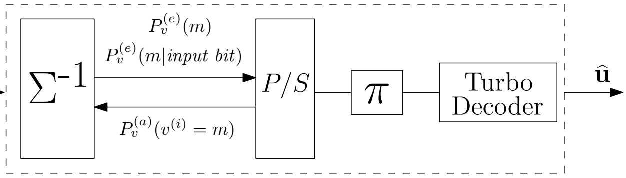

Fig. 2. Decoding scheme.

For all $V^{(i)}$ corresponding to systematic bits, the *extrinsic* probability $P_{v^{(i)}}^{(e)}(m)$ generated by the sigma-demapper is proportional to

$$P_{v^{(i)}}^{(e)}(m) \propto P_v^{(initial)}(v^{(i)} = m) \sum_{\mathbf{v} \in \mathbb{F}_2^k} \mathbb{I}(v^{(i)} = m) P_{\mathbf{V}}(\mathbf{v}|v^{(i)}) P_{Y|V}(y|\phi(\mathbf{s})) \qquad (2)$$

where $\mathbb{F}_2^k$ denotes the k-extension of the binary Galois Field, and $\mathbb{I}(P)$ is the indicator function which takes the value 1 when the proposition P is true and 0 otherwise. In the above expression, $P_{\mathbf{V}}(\mathbf{v}|v^{(i)})$ is given by

$$P_{\mathbf{V}}(\mathbf{v}|v^{(i)}) = \prod_{j \neq i} P_v^{(initial)}(v^{(j)}) P_v^{(a)}(v^{(j)}), \qquad (3)$$

whereas $P_{Y|V}(y|\phi(\mathbf{s}))$ is proportional to

$$P_{Y|V}(y|\phi(\mathbf{s})) \propto \sum \exp\left(-\frac{(y - \phi(\mathbf{s}))^2}{2\sigma^2}\right), \qquad (4)$$

where the number of elements in the sum varies depending on the number of parity and systematic bits that compose the vector $\mathbf{v}$. The reason being that for a given value of a systematic bit, there is only one possible value of its associated modulated symbol s while for a parity bit, two values of s are possible depending on the value of the associated input bit. For example, consider the case where $k = 2$ and $(V^{(1)}, V^{(2)}) = (0, 0)$. If $V^{(1)}$ and $V^{(2)}$ are both systematic, the value of $\phi(\mathbf{s})$ is unique and equal to $-2\sqrt{p_1/p_0}$. On the contrary, if one is systematic and the other a parity bit, two values of $\phi(\mathbf{s})$ are possible, namely, $-\sqrt{p_1/p_0} - \sqrt{\theta/p_0}$ and $-\sqrt{p_1/p_0} - \sqrt{(1-\theta)/p_0}$. Finally, when $V^{(1)}$ and $V^{(2)}$ are both parity bits, $\phi(\mathbf{s})$ may take four different values. Besides, when the vector $\mathbf{v}$ contains parity bits, the exponential term of the righthand side of expression (4) has to be multiplied by the *a priori* probabilities of the input bits associated with such parity bits. When calculating the *extrinsic* probabilities $P_v^{(e)}(m|\textit{input bit})$ of the parity bits, the only difference is that in expression (4) we do not have to multiply the exponential term by the *a priori* probability of the input bit associated with the parity bit we are considering. Furthermore notice that expression (2) is not multiplied by $P_v^{(a)}(v^{(i)} = m)$ to avoid any positive feedback to the Turbo decoder.

Once computed by the sigma-demapper, all the above *extrinsic* probabilities are passed to the Turbo decoder through a parallel-to-serial converter and the deinterleaver π^{-1}. The Turbo decoder employs these probabilities as *a priori* probabilities, and runs the SPA algorithm over the FG that describes its compounding RSC codes. However, note that a slight modification is required in such FG, since in our scheme *extrinsic* information for the parity bits is also needed to

---

[2] It has been observed that the use of Type-II sigma-mapper does not improve the performance of our scheme due to the utilized unequal energy allocation.

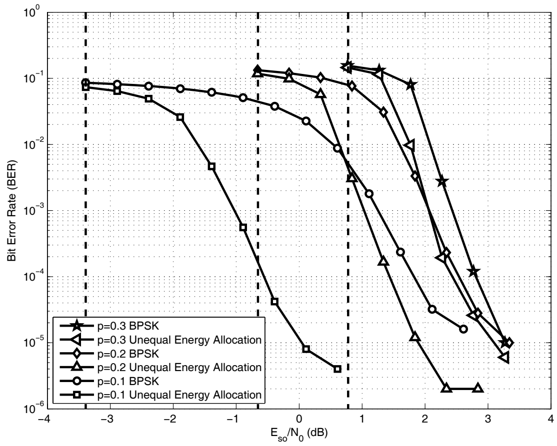

Fig. 3. BER versus E_{so}/N_0 for $p_0 \in \{0.1, 0.2, 0.3\}$.

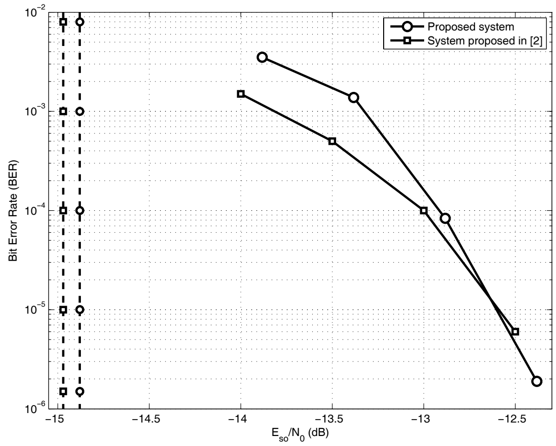

Fig. 4. BER versus E_{so}/N_0 for $p_0 = 0.005$.

be passed to the sigma-demapper. The *extrinsic* probabilities generated by the Turbo decoder are then used as *a priori* probabilities by the sigma-demapper. The decoder stops after a fixed number of iterations.

IV. SIMULATION RESULTS

In order to verify the performance of the proposed scheme, two sets of simulations have been performed. The goal of the first set is to asses the improvement that the unequal energy allocation strategy entails in the overall performance of the proposed scheme. To that end, the proposed system is simulated with and without energy allocation. In the latter case the encoded symbols at the output of the Turbo coder are simply BPSK modulated (i.e. by using equal energy). The Turbo code from [6, Example A] with rate $R_c = 1/3$ and generator polynomial $G(D) = 1/(1 + D)$ has been used. The value of k was set to $k = 3$, so the transmission rate R is 2 binary source symbol per 2D (bandwidth-limited regime).

Monte Carlo simulations have been performed for a block-length $K = 10000$ and a maximum of 50 decoding iterations. Three source symbol distributions $p_0 \in \{0.1, 0.2, 0.3\}$ have been selected, giving rise to source entropies $H(p_0) = 0.47$, 0.72 and 0.88 bits per source symbol, respectively. Figure 3 shows the BER versus E_{so}/N_0, where the vertical lines correspond to the theoretical limit given by expression (1). Notice that for $p_0 = 0.1$, the performance improvement when utilizing, in our scheme, an unequal energy allocation modulation instead of standard BPSK is 2.37 dB at a BER = 10^{-4}. On the other hand, observe that the proposed scheme is 2.84 dB away from the Shannon limit. Further simulations included in the plot show that, for $p_0 = 0.2$ and $p_0 = 0.3$, we obtain a gain of 1.1 and 0.37 dB, respectively, and they are 2.08 and 1.66 dB away from their theoretical limit.

The above gaps to the Shannon limit can be reduced by using a more powerful Turbo code. This is shown in the second set of simulations, where the goal is also to compare our scheme operating at the bandwidth-limited regime with the Turbo coding scheme proposed in [2] (suitable for operating only at the power-limited regime). In this comparison, both systems use the same Turbo code of rate $R_c = 1/3$, generator polynomial $G(D) = (1 + D + D^2 + D^4)/(1 + D^3 + D^4)$ and $K = 16384$. Notice that although both schemes utilize the same Turbo code of rate $R_c = 1/3$, our scheme implements a transmission rate of 2 ($R = 2$), whereas the scheme in [2] one of $2/3$ ($R = 2/3$). Figure 4 plots the BER versus E_{so}/N_0 for $p_0 = 0.005$, where the vertical lines represent the corresponding theoretical limits of both systems. Although both schemes need approximately the same E_{so}/N_0 to obtain a BER of 10^{-5}, our system performs slightly better, since its corresponding Shannon limit is closer. Further results show that for $p_0 = 0.01$, both schemes are 2.0 dB away of their corresponding theoretical limits.

V. CONCLUSION

We have proposed a one-layer coding/shaping system with single-level codes and sigma-mapping, for the case of non-uniform memoryless sources sent over AWGN channels. The proposed scheme has proved to have similar or slightly better BER performance, in the bandwidth-limited regime, when compared to the system in [2] for the power-limited regime.

ACKNOWLEDGMENT

This work was supported in part by the Spanish Ministry of Science and Innovation through the CONSOLIDER-INGENIO 2010 programme (CSD200800010, www.comonsens.org).

REFERENCES

[1] F. Cabarcas, R. D. Souza, and J. Garcia-Frias, “Source-controlled turbo coding of non-uniform memoryless sources based on unequal energy allocation,” in *Proc. IEEE International Symposium on Information Theory*, p. 166, June 2004.

[2] F. Cabarcas, R. D. Souza, and J. Garcia-Frias, “Turbo coding of strongly non-uniform memoryless sources with unequal energy allocation and PAM signaling,” *IEEE Trans. Signal Process.*, vol. 54, no. 5, pp. 1942-1946, May 2006.

[3] G. I. Shamir and J. J. Boutros, “Non-systematic low-density parity-check codes for nonuniform sources,” in *Proc. 2005 IEEE International Symposium on Information Theory*, pp. 1898-1902, Sep. 2005.

[4] R. D. Souza, G. I. Shamir, J. Garcia-Frias, and K. Xie, “Non-systematic turbo coding with unequal energy allocation for nonuniform memoryless sources,” in *Proc. IEEE International Symposium on Information Theory*, pp. 1893-1897, Sep. 2005.

[5] G. Zhu, F. Alajaji, J. Bajcsy, and P. Mitran, “Non-systematic turbo codes for non-uniform i.i.d. sources over AWGN channels,” *Conference on Information Sciences and Systems*, Mar. 2002.

[6] X. Ma and L. Ping, “Coded modulation using superimposed binary codes,” *IEEE Trans. Inf. Theory*, vol. 50, no. 12, Dec. 2004.